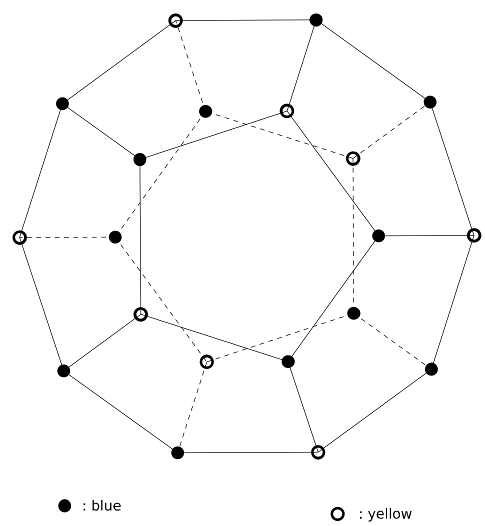
We show that the maximal number of blue faces is 12. For convenience, we shall use the dodecahedral graph $\mathcal{D}$ whose vertices correspond to the faces of the icosahedron, two vertices being linked by an edge when the corresponding faces are adjacent. A coloring in blue or yellow of the faces of the icosahedron corresponds to a coloring in blue or yellow of the vertices of $\mathcal{D}$. Each of the 20 vertices of $\mathcal{D}$ has degree 3 (see figure) and the coloring will respect the constraint of the problem if and only if at most one blue vertex is adjacent to each blue vertex. A suitable coloring with 12 blue vertices is presented on the figure.



Now, we show that a suitable coloring cannot have more than 12 blue vertices. Each of the twenty faces of the dodecahedron has five vertices of which three at most are blue vertices (otherwise at most one vertex would be yellow so that a blue vertex would necessarily be adjacent to two blue vertices). If we count the blue vertices by adding the blue vertices obtained face after face, this provides a totality of at most 3×12 blue vertices. However, each vertex of $\mathcal{D}$ is a vertex of exactly three faces, hence each of the blue vertices counted just before is counted three times. Thus, we actually have at most 12 blue vertices.

4103. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Daniel Sitaru.*

Let x, y and z be positive numbers such that $x + y + z = 1$. Show that

$$\sum_{\text{cyc}} [(1-x)\sqrt{3yz(1-y)(1-z)}] \geq 4\sqrt{xyz}.$$

There were six correct solutions. Three of the solutions used algebraic inequalities while the remaining three employed trigonometry. We present three different solutions here.

Solution 1, by Kee-Wai Lau.

The inequality is equivalent to

$$\sum \left[(1-x)\sqrt{\frac{x+yz}{x}}\right] \geq \frac{4}{\sqrt{3}},$$

or

$$\sum \sqrt{1+\frac{yz}{x}} \geq \frac{4}{\sqrt{3}} + \sum \sqrt{x(x+yz)},$$

where each sum is cyclic with three terms. Thus it suffices to show that

$$\sum \sqrt{1+\frac{yz}{x}} \geq 2\sqrt{3} \tag{1}$$

and

$$\sum \sqrt{x(x+yz)} \leq \frac{2}{\sqrt{3}}. \tag{2}$$

For (1), we have that

$$\begin{aligned}
&\left(\sum \sqrt{1+\frac{yz}{x}}\right)^2 \\
&= \sum \left(1+\frac{yz}{x}\right) + 2\sum \sqrt{\left(1+\frac{yz}{x}\right)\left(1+\frac{zx}{y}\right)} \\
&= 3 + \sum \frac{yz}{x} + 2\left(\sum \sqrt{(1+z)^2 + \frac{z(x-y)^2}{xy}}\right) \\
&= 3 + \left(\sum x + \frac{1}{2xyz}\sum x^2(y-z)^2\right) + 2\left(\sum \sqrt{(1+z)^2 + \frac{z(x-y)^2}{xy}}\right) \\
&\geq 4 + 2\sum(1+z) \\
&= 12.
\end{aligned}$$

For (2), the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
\left(\sum \sqrt{x(x+yz)}\right)^2 &\leq \left(\sum x\right)\left(\sum (x+yz)\right) \\
&= 1 + \sum yz \\
&= 1 + \frac{1}{6}\left[2(x+y+z)^2 - (x-y)^2 - (y-z)^2 - (z-x)^2\right] \\
&\leq 1 + \frac{1}{6}\cdot 2 = \frac{4}{3}.
\end{aligned}$$

The result follows, with equality occurring iff $x=y=z=1/3$.

Solution 2, by AN-anduud Problem Solving Group.

There exists a triangle ABC with sides of lengths $a = y + z$, $b = z + x$, $c = x + y$. Let R be its circumradius and s its semi-perimeter; $s = x + y + z = 1$. The inequality is equivalent to

$$\sqrt{3}\left[a\sqrt{\frac{bc}{s(s-a)}} + b\sqrt{\frac{ca}{s(s-b)}} + c\sqrt{\frac{ab}{s(s-c}}\right] \geq 2(a+b+c).$$

Noting that $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $a = 2R\sin A = 4R\cos\frac{A}{2}\sin\frac{A}{2}$, *etc.*, we have to establish that

$$4\sqrt{3}R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \geq 8R\left(\cos\frac{A}{2}\sin\frac{A}{2} + \cos\frac{B}{2}\sin\frac{B}{2} + \cos\frac{C}{2}\sin\frac{C}{2}\right).$$

Recall that

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}$$

(by Jensen's Theorem, for example) and that

$$3(u_1v_1 + u_2v_2 + u_3v_3) \leq (u_1 + u_2 + u_3)(v_1 + v_2 + v_3)$$

when $u_1 \geq u_2 \geq u_3 > 0$ and $0 < v_1 \leq v_2 \leq v_3$ (Chebyshev's sum inequality). Since the cosine and sine functions are monotonely opposite on $(0, \pi/2)$.

$$\begin{aligned}
&8R(\cos\frac{A}{2}\sin\frac{A}{2} + \cos\frac{B}{2}\sin\frac{B}{2} + \cos\frac{C}{2}\sin\frac{C}{2}) \\
&\leq \frac{8R}{3}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right) \\
&\leq 4\sqrt{3}R\left(\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}\right),
\end{aligned}$$

as desired.

Solution 3, by Digby Smith.

We first establish that, for $x, y, z > 0$,

$$\sum \frac{x(y+z)}{(x+y)(x+z)} \leq \frac{3}{2},$$

with equality iff $x = y = z$. (The sum is cyclic with three terms.) This follows from

$$\begin{aligned}
&3(x+y)(y+z)(z+x) - 2[x(y+z)^2 + y(z+x)^2 + z(x+y)^2] \\
&= 3[x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2) + 2xyz] \\
&\quad - 2[x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2) + 6xyz] \\
&= x(y^2+z^2) + y(z^2+x^2) + z(x^2+y^2) - 6xyz \\
&\geq 2xyz + 2xyz + 2xyz - 6xyz = 0.
\end{aligned}$$

Because $x + y + z = 1$, the inequality is equivalent to

$$\left(\sum (y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^2 \geq \frac{16}{3}.$$

Recall the Hölder inequality with u_i and v_i nonnegative for $i = 1, 2, 3$:

$$\left(\sum_{i=1}^{3} u_i^3\right)^{1/3} \left(\sum_{i=1}^{3} v_i^{3/2}\right)^{2/3} \geq \sum_{i=1}^{3} u_i v_i.$$

Applying this to the triples

$$(u_1, u_2, u_3) = \left(\left(\frac{x(y+z)}{(x+y)(x+z)}\right)^{1/3}, \left(\frac{y(z+x)}{(y+z)(y+x)}\right)^{1/3}, \left(\frac{z(x+y)}{(z+x)(z+y)}\right)^{1/3}\right)$$

and (v_1, v_2, v_3)

$$= \left(\left((y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^{\frac{2}{3}}, \left((z+x)\sqrt{\frac{(y+z)(y+x)}{y}}\right)^{\frac{2}{3}}, \left((x+y)\sqrt{\frac{(z+x)(z+y)}{z}}\right)^{\frac{2}{3}}\right),$$

and using the preliminary result, we find that

$$\begin{aligned}
&\frac{3}{2}\left(\sum (y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^2 \\
&\geq \left(\sum \frac{x(y+z)}{(x+y)(x+z)}\right)\left(\sum (y+z)\sqrt{\frac{(x+y)(x+z)}{x}}\right)^2 \\
&\geq \left(\sum (y+z)\right)^3 = 8.
\end{aligned}$$

The desired inequality follows directly.

4104. *Proposed by Daniel Sitaru.*

Prove that for $0 < a \leq b \leq c \leq d < 2$, we have

$$5(ab^4 + bc^4 + cd^4 + 16d) < 5(b^5 + c^5 + d^5 + 16a) + 128.$$

There were five correct solutions. We present two different ones here.

Solution 1, by Šefket Arslanagić; and Salem Malikić (independently).

By the arithmetic-geometric means inequality, we have

$$a^5 + 4b^5 \geq 5ab^4, \quad b^5 + 4c^5 \geq 5bc^4, \quad c^5 + 4d^5 \geq 5cd^4$$

and

$$d^5 + 128 = d^5 + 4 \cdot 2^5 \geq 5 \cdot 2^4 d = 80d.$$